

A sharp maximal inequality for differentially subordinate martingales under a change of law.

Komla Domelevo Stefanie Petermichl

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Abstract

We prove a sharp weighted L^p estimate of Y^* with respect to X . Here Y and X are uniformly integrable càdlàg Hilbert space valued martingales and Y differentially subordinate to X via the square bracket process. The proof is via an iterated stopping procedure and self similarity argument known as 'sparse domination'. We point out that in this generality, the special case $Y = X$ addresses a question raised by Bonami–Lépingle. The proof via sparse domination for the latter spacial case has optimal bounds in terms of the necessary characteristic of the weight when $1 < p \leq 2$ only. The case for $p > 2$ is addressed by other means.

1 Introduction

We have a filtered probability space with the usual assumptions. Let X and Y be adapted uniformly integrable càdlàg martingales. Burkholder's definition of differential subordination has appeared as the correct continuous in time replacement of the corresponding notion in probability spaces with discrete filtration and predictable multipliers. We say that Y is differentially subordinate to X if and only if $[X, X]_t - [Y, Y]_t$ is non-negative and non-decreasing. Here $[\cdot, \cdot]$ denotes the quadratic variation process. We prove the following sharp estimate

$$\|Y^*\|_{L^p(w)} \leq c_p Q_p(w)^{\max(1, 1/(p-1))} \|X\|_{L^p(w)}$$

where w is a positive uniformly integrable martingale called a weight. The quantity below is the A_p characteristic of the weight w :

$$Q_p(w) = \sup_{\tau} \left\| \mathbb{E} \left[\left(\frac{w_{\tau}}{w} \right)^{1/p-1} \middle| \mathcal{F}_{\tau} \right]^{p-1} \right\|_{\infty} = \sup_{\tau} \|w_{\tau} u_{\tau}^{p-1}\|_{\infty}$$

where τ is an adapted stopping time and where we write $w = w_{\infty}$ and $u^p w = u$.

This and similar lines of questions has a long history and has proven to be influential in the weighted theory in harmonic analysis. Weighted norm

estimates for typical classical operators often follow similar patterns as their probabilistic models. Certain classical operators in harmonic analysis, such as Riesz transforms can be written as a conditional expectation of certain martingale transforms [8]. Other, deeper connections have surfaced in the last ten to twenty years [17][9]. During the last twenty years interest has shifted towards sharp norm estimates in terms of the characteristic of the weight, in part thanks to the solution of a long standing regularity problem in PDE through a sharp weighted norm estimate of the Beurling operator [18].

One of the first sharp estimates however, was on predictable multipliers for dyadic filtrations in the interval $[0, 1]$ endowed with Lebesgue measure [22], relying on Bellman functions and methods developed in [15]. Meanwhile a number of other, beautiful proofs have appeared, for example [12][4]. Its extension to predictable multipliers in general discretely filtered spaces is subtle and very recent [20] using a combination of outer measure space technique and some 'smaller' Bellman functions. The solution of the general case, where filtrations have a continuous parameter and the martingales are càdlàg and merely in relation of differential subordination using the square bracket is by the authors in [7]. The proof is technical and has required a 'large' Bellman function with subtle convexity properties, solving the entire problem at once and using an array of techniques and tricks developed in recent years.

All of these results merely estimate Y , not Y^* , the maximal function of Y . The first result of this stronger nature is very recent: the sharp weighted norm estimate of the maximal function of predictable multipliers in the discrete in time filtration case is due to Lacey [11]. It is the main result in this present paper to prove that the estimate holds for general pairs of martingales under differential subordination in general filtered spaces, thus removing the restrictions of a discretely filtered space as well as the predictable multiplier property. This approach extends the technique of sparse forms and their weighted bounds to continuous stopping times. We give a precise definition of a sparse form in this setting.

The special case $Y = X$ becomes a weighted estimate for the maximal function of X . The first such estimate for the dyadic filtration was proved by Muckenhoupt [14]. It was brought to the probabilistic context by Izumisawa and Kazamaki [10] for continuous in space martingales. Later, Buckley [3] improved the dependence on the weight's characteristic in the dyadic setting and Osekowski [16] provided the proof for continuous in space martingales.

The continuity assumption on the path of the martingales is a notable restriction. In the weighted case, useful facts no longer hold when this assumption is dropped. One of these is the self improvement property of the A_p classes, which is used in these classical proofs. It is an old observation that this fact is false for weights in general filtrations allowing jumps [1] and an additional homogeneity type condition controlling the jumps was assumed by Doleans-Dade-Meyer [6] to confirm a weighted maximal inequality. Bonami-Lépingle [2] observed that for the example weight that belongs to A_p but not $A_{p-\varepsilon}$ (i.e. self improvement fails), the weighted maximal inequality still holds and phrased the question

for the general filtrations allowing jumps. The positive answer follows for all $1 < p < \infty$ as a special case of our main result. It is however only sharp in dependence on the A_p characteristic when $p \leq 2$. For this reason, we give another proof for the special case $Y = X$ that gives the best estimate for all p . Indeed we prove that

$$\|X^*\|_{L^p(w)} \leq c_p Q_p(w)^{1/(p-1)} \|X\|_{L^p(w)}.$$

The proof yielding the estimate for Y^* uses a technique called sparse domination, by now well established in the weighted theory in harmonic analysis and has proven very powerful, especially with recent developments, where the domination is pointwise. The idea is to construct self similar operators through a stopping time procedure where the consecutive stopping time is finite for a small portion of the underlying probability space (more has to hold, see the precise definition of sparse form below). This gives rise to a pointwise domination of Y^* by a positive form containing $|X|$ sampled at various consecutive stopping times with some additional features. The latter representation lends itself well to an estimate in weighted space through a change of measure and the use of Doob's inequality.

The proof for the estimate of X^* uses a very simple domination of the maximal function, direct, without the use of stopping times. This approach is a modification of a trick due to Lerner [13], where it was used in a different context. It also allows the use of Doob's inequality and a change of measure to give the desired result.

Notice that maximal inequalities involving X^* can be deduced from the discrete in time filtered general case through the use of Doob's sampling theorem, but not the estimates involving Y^* , since differential subordination is not preserved when sampling the martingale.

2 Definitions and Main Results

Let $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ be a complete filtered probability space with $\mathfrak{F} = (\mathcal{F}_t)_{t \geq 0}$ a filtration that is right continuous, where \mathcal{F}_0 contains all \mathcal{F} null sets. Let X and Y be uniformly integrable adapted càdlàg martingales with values in a separable Hilbert space that are in a relation of differential subordination according to Burkholder:

Definition 1 *Y is called differentially subordinate to X if $[X, X]_t - [Y, Y]_t$ is nonnegative and nondecreasing in t .*

For the definition of the square bracket process and its properties, see for example Dellacherie–Meyer [5] or Protter [19]. Notice that in particular $[X, X]_0 = |X_0|^2$ so that differential subordination of Y with respect to X implies $|Y_0|^2 \leq |X_0|^2$.

Recall that by $X^* = \sup_{t \geq 0} |X_t|$ we denote the maximal function associated with X . We shall throughout this text denote by the same letters also the closures of the martingales that arise.

Here is our main theorem.

Theorem 1 *Y differentially subordinate to X then*

$$\|Y^*\|_{L^p(w)} \leq 16 \frac{p^2}{p-1} Q_p(w)^{\max(1, 1/(p-1))} \|X\|_{L^p(w)}.$$

In the special case $Y = X$ we also prove

Theorem 2

$$\|X^*\|_{L^p(w)} \leq \frac{p^{p'}}{p-1} Q_p(w)^{1/(p-1)} \|X\|_{L^p(w)}.$$

Both estimates are well known to be sharp in terms of the dependence on the A_p characteristic, already for dyadic filtration on $[0, 1]$ endowed with Lebesgue measure.

3 Maximal Function of Y

In this section we prove Theorem 1.

3.1 Sparse Decomposition

We will use the following preliminary unweighted result due to Wang [21], see Theorem 3. (Wang's result is slightly stronger, but the above is all we will need)

Theorem 3 *Y differentially subordinate to X then*

$$\mathbb{P}(\{\omega \in \Omega : Y^*(\omega) \vee X^*(\omega) > \lambda\}) \leq 2\lambda^{-1} \|X\|_1.$$

First let $\mathfrak{F}^0 = \mathfrak{F}$ and $\mathcal{F}_t^0 = \mathcal{F}_t$ and consider the martingale $Y_t^0 = Y_t/|X|_0$, still differentially subordinate to $X_t^0 = X_t/|X|_0$. The set

$$E_0 = \{\omega \in \Omega : Y^{0*}(\omega) \vee X^{0*}(\omega) > 4\}$$

has small measure, thanks to Theorem 3

$$\mathbb{P}(E_0) \leq \frac{2}{4} \|X^0\|_1 = \frac{2}{4} \mathbb{E} \left[\frac{\mathbb{E}[|X| | \mathcal{F}_0]}{|X|_0} \right] = \frac{1}{2}.$$

There is associated a hitting time $T^0(\omega)$ of the set $L = (4, \infty)$, that is

$$T^0(\omega) = \inf\{t > 0 : Y_t^0(\omega) \vee X_t^0(\omega) \in L\}.$$

$T^0(\omega)$ is a stopping time and finite in E_0 almost surely.

By differential subordination almost surely $|Y_{T^0}(\omega) - Y_{T_-^0}(\omega)| \leq |X_{T^0}(\omega) - X_{T_-^0}(\omega)|$. Thus, there exists linear operator $r(\omega) \in \mathcal{F}_{T^0}$ such that $|r| \leq 1$ and $Y_{T^0}(\omega) - Y_{T_-^0}(\omega) = r(\omega)(X_{T^0}(\omega) - X_{T_-^0}(\omega))$. Choose $r = 0$ when $Y_{T^0}(\omega) = Y_{T_-^0}(\omega)$ such as when there is no jump.

As usual let the stopping sigma algebra be $\mathcal{F}_{T^0} = \{\Lambda \in \mathcal{F} : \Lambda \cap \{T^0 \leq t\} \in \mathcal{F}_t\}$. Let the filtration $\mathfrak{F}^1 = (\mathcal{F}_t^1)_{t \geq 0} = (\mathcal{F}_{T^0 \vee t})_{t \geq 0}$. Observe that $E_0 \in \mathcal{F}_{T^0}$. Consider

$$Y_t^1 = \chi_{E_0}(\mathbb{E}[Y|\mathcal{F}_t^1] - Y_{T^0} + rX_{T^0})/|X|_{T^0} = \chi_{E_0} \left(rX_{T^0} + \int_{T_+^0}^{t \vee T^0} dY_u \right) / |X|_{T^0}$$

and

$$X_t^1 = \chi_{E_0} \mathbb{E}[X|\mathcal{F}_t^1] / |X|_{T^0}.$$

These are martingales with respect to \mathcal{F}_t^1 and Y^1 is differentially subordinate to X^1 . To see this, one uses measurability of E_0 in \mathcal{F}_0^1 and $[X^\tau, X^\tau] = [X, X]^\tau$ for any stopping time τ .

There holds almost surely

$$Y^* \leq 8|X|_0 + |X|_{T^0} Y^{1*}.$$

Indeed, when $T^0 = \infty$ then the stronger inequality $Y^* \leq 4|X|_0$ holds by the definition of T^0 . Otherwise and when $t < T^0$ then $|Y_t| \leq 4|X|_0$. When $t \geq T^0$ write

$$Y_t = \left(Y_0 + \int_{0+}^{T_-^0} dY_u \right) - (rX_{T_-^0}) + \left(rX_{T^0} + \int_{T_+^0}^t dY_u \right).$$

The first two summands are each controlled by $4|X|_0$ by the definition of the stopping time. So it remains to observe that the last term on E_0 is dominated by $|X|_{T^0} Y^{1*}$.

Next we obtain a subset of E_0 :

$$E_1 = \{\omega \in \Omega : Y_t^{1*}(\omega) \vee X_t^{1*}(\omega) > 4\},$$

a stopping time $T^1 \geq T^0$ and filtration $\mathfrak{F}^2 = (\mathcal{F}_t^2)_{t \geq 0} = (\mathcal{F}_{T^1 \vee t}^1)_{t \geq 0}$. Build Y_t^2 and X_t^2 as before. Iterating we therefore obtain

$$Y^*(\omega) \leq 8|X|_0(\omega) + 8|X|_{T^0}(\omega)\chi_{E_0}(\omega) + 8|X|_{T^1}(\omega)\chi_{E_1}(\omega) + \dots$$

When writing $T^{-1} = 0$ and $E_{-1} = \Omega$ we have

$$Y^*(\omega) \leq 8 \sum_{j=-1}^{\infty} |X|_{T^j}(\omega)\chi_{E_j}(\omega)$$

Observe that when $A^j \subset E_j$ measurable in $\mathcal{F}_{T^j} = \mathcal{F}_0^{j+1}$ then $Y_t^{j+1}\chi_{A^j}$ and $X_t^{j+1}\chi_{A^j}$ remain differentially subordinate martingales in $\mathfrak{F}^{j+1} = (\mathcal{F}_{T^j \vee t})_{t \geq 0}$. Their weak type bounds give us in the next step

$$\mathbb{P}(A^j \cap E_{j+1}) = \mathbb{P}(\{\omega \in \Omega : (Y_t^{j+1}\chi_{A^{(j)}})^* \vee (X_t^{j+1}\chi_{A^{(j)}})^* > 4\}) \leq \frac{1}{2}\mathbb{P}(A^j)$$

This last observation gives local control and is an important feature.

3.2 Estimate of the Sparse form

We are led to make the following definition:

Definition 2 Let $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ as above and X uniformly integrable adapted càdlàg martingale with values in a separable Hilbert space. The operator $X \mapsto \mathcal{S}(|X|)$ is called sparse if there exists an increasing sequence of adapted stopping times $0 = T^{-1} \leq T^0 \leq \dots$ with nested sets $E_j = \{T^j < \infty\}$ so that

- i. $\mathcal{S}(|X|) = \sum_{j=-1}^{\infty} |X|_{T^j}(\omega) \chi_{E_j}(\omega)$
- ii. $\forall A^j \subset E_j, A^j \in \mathcal{F}_{T^j}$ there holds $\mathbb{P}(A^j \cap E_{j+1}) \leq \frac{1}{2}\mathbb{P}(A^j)$.

In the previous section we have seen that Y^* is dominated almost surely by a multiple of a sparse operator. In order to prove our main theorem for the case $p = 2$, it suffices to show that

$$\|\mathcal{S}(|\tilde{X}|)\|_{L^2(w)} \leq c_2 Q_2(w) \|\tilde{X}\|_{L^2(w)}.$$

This means (suppressing the ∞ subscripts as usual)

$$\mathbb{E}[(\mathcal{S}(|\tilde{X}|))^2 w] \leq c_2 Q_2(w) \mathbb{E}[\tilde{X}^2 w].$$

Dualizing we get the required estimate

$$\mathbb{E}[\mathcal{S}(|\tilde{X}|)|\tilde{Z}] \leq c_2 Q_2(w) \mathbb{E}(|\tilde{X}|^2 w)^{1/2} \mathbb{E}(|\tilde{Z}|^2 w)^{1/2}$$

Now write $|X|u = |\tilde{X}|$ and $|Z|w = |\tilde{Z}|$ it suffices to prove

$$\mathbb{E}[\mathcal{S}(|X|u)|Z|w] \leq c_2 Q_2(w) \mathbb{E}[w]^{1/2} \mathbb{E}[u]^{1/2} \mathbb{E}_u[|X|^2]^{1/2} \mathbb{E}_w[|Z|^2]^{1/2}.$$

Now, we calculate the left hand side

$$\begin{aligned} \mathbb{E} \left[\sum_j (|X|u)_{T^j} \chi_{E_j} |Z|w \right] &= \mathbb{E} \left[\sum_j \mathbb{E}[(|X|u)_{T^j} \chi_{E_j} |Z|w | \mathcal{F}_{T^j}] \right] \\ &= \mathbb{E} \left[\sum_j (|X|u)_{T^j} (|Z|w)_{T^j} \chi_{E_j} \right] \end{aligned}$$

Now use that

$$(|Z|w)_{T^j} \chi_{E_j} = \mathbb{E}[|Z|w | \mathcal{F}_{T^j}] \chi_{E_j} = \mathbb{E}_w[|Z| | \mathcal{F}_{T^j}] \mathbb{E}[w | \mathcal{F}_{T^j}] \chi_{E_j}$$

and similarly for the other term, recalling that by the A_2 condition

$$\|\mathbb{E}[w | \mathcal{F}_{T^j}] \mathbb{E}[w^{-1} | \mathcal{F}_{T^j}]\|_\infty \leq Q_2(w),$$

the above is bounded by

$$Q_2(w) \mathbb{E} \left[\sum_j |X|_{T^j,u} |Z|_{T^j,w} \chi_{E_j} \right]$$

if we write for instance $|X|_{T^j,u} = \mathbb{E}_u[|X| | \mathcal{F}_{T^j}]$.

Now for each fixed j we have that the non-negative random variable

$$|X|_{T^j,u} |Z|_{T^j,w} \chi_{E_j} \in \mathcal{F}_{T^j}$$

and as such it can be approximated from below by step functions.

$$\sum_k \alpha_k^j \chi_{A_k^j} \nearrow |X|_{T^j,u} |Z|_{T^j,w} \chi_{E_j}$$

with $A_k^j \in \mathcal{F}_{T^j}$ disjoint and $\cup A_k^j = E_j$. Notice that on A_k^j there holds

$$\alpha_k^j \leq X_{j,u}^* Z_{j,w}^*$$

where the maximal functions are taken with respect to filtration \mathfrak{F}^{j+1} with weighted measure. Observe that everywhere

$$X_{j,u}^* Z_{j,w}^* \leq X_u^* Z_w^*$$

where on the right hand side the maximal function is taken with respect to the original filtration but weighted measure.

Now recall that $\mathbb{P}(A_k^j \cap E_{j+1}) \leq \frac{1}{2} \mathbb{P}(A_k^j)$ and so $\mathbb{P}(A_k^j \setminus (A_k^j \cap E_{j+1})) \geq \frac{1}{2} \mathbb{P}(A_k^j)$. Thus

$$\mathbb{E} \left[\sum_k \alpha_k^j \chi_{A_k^j} \right] \leq 2 \sum_k \alpha_k^j \mathbb{P}(A_k^j \setminus (A_k^j \cap E_{j+1})).$$

Sets $S_k^j = A_k^j \setminus (A_k^j \cap E_{j+1})$ are disjoint in both parameters. Now we use this fact to bound the term

$$\mathbb{E} \left[\sum_j |X|_{T^j,u} |Z|_{T^j,w} \chi_{E_j} \right].$$

By the monotone convergence theorem for every j fixed

$$\mathbb{E} \left[\sum_k \alpha_k^j \chi_{A_k^j} \right] \nearrow \mathbb{E}[|X|_{T^j,u} |Z|_{T^j,w} \chi_{E_j}].$$

We have for finite sums

$$\begin{aligned} \mathbb{E} \left[\sum_{j=-1}^J \sum_k \alpha_k^j \chi_{A_k^j} \right] &= \sum_{j=-1}^J \sum_k \alpha_k^j \mathbb{E}[\chi_{A_k^j}] \\ &= \sum_{j=-1}^J \sum_k \alpha_k^j \mathbb{P}(A_k^j) \\ &\leq 2 \sum_{j=-1}^J \sum_k \alpha_k^j \mathbb{P}(S_k^j) \\ &= 2 \mathbb{E} \left[\sum_{j=-1}^J \sum_k \alpha_k^j \chi_{S_k^j} \right] \\ &\stackrel{(1)}{=} 2 \mathbb{E} \left[\sum_{j=-1}^J \sum_k \alpha_k^j \chi_{S_k^j} w^{1/2} u^{1/2} \right] \\ &\leq 2 \left(\mathbb{E} \left[\sum_{j=-1}^J \sum_k (X_{j,u}^*)^2 u \chi_{S_k^j} \right] \right)^{1/2} \times \\ &\quad \times \left(\mathbb{E} \left[\sum_{j=-1}^J \sum_k (Z_{j,w}^*)^2 w \chi_{S_k^j} \right] \right)^{1/2} \\ &\leq 2(\mathbb{E}[(X_u^*)^2 u])^{1/2} (\mathbb{E}[(Z_w^*)^2 w])^{1/2} \\ &= 2(\mathbb{E}[u])^{1/2} (\mathbb{E}[w])^{1/2} (\mathbb{E}_u[(X_u^*)^2])^{1/2} (\mathbb{E}_w[(Z_w^*)^2])^{1/2} \\ &\leq 8(\mathbb{E}[w])^{1/2} (\mathbb{E}[u])^{1/2} (\mathbb{E}_u[|X|^2])^{1/2} (\mathbb{E}_w[|Z|^2])^{1/2} \end{aligned}$$

This gives us the desired estimate

$$\mathbb{E} \left[\sum_j |X|_{T^j,u} |Z|_{T^j,w} \chi_{E_j} \right] \leq 8 \mathbb{E}[w]^{1/2} \mathbb{E}[u]^{1/2} \mathbb{E}_u[|X|^2]^{1/2} \mathbb{E}_w[|Z|^2]^{1/2}.$$

We now point out the changes for the case $p \neq 2$. We set u the dual weight of w in the sense $u^p w = u$. So $\|\mathcal{S}|X|\|_{L^p(w)} \leq c_p Q_p(w)^{\max(1, 1/p-1)} \|X\|_{L^p(w)}$ becomes

$$\begin{aligned} \|\mathcal{S}|X|u\|_{L^p(w)} &\leq c_p Q_p(w)^{\max(1, 1/(p-1))} \|X|u\|_{L^p(w)} \\ &= c_p Q_p(w)^{\max(1, 1/(p-1))} \|X\|_{L^p(u)}. \end{aligned}$$

As before we set up by duality and calculate

$$\mathbb{E} \left[\sum_j (|X|u)_{T^j} \chi_{E_j} |Z|w \right] = \mathbb{E} \left[\sum_j |X|_{T^j,u} |Z|_{T^j,w} \chi_{E_j} u_{T^j} w_{T^j} \right]$$

Writing $p^* = \max(p, p')$, notice that we aim at a constant

$$Q_p(w)^{\max(1, 1/(p-1))} = \sup_{\tau} \|w_{\tau}^{p^*/p} u_{\tau}^{p^*/p'}\|_{\infty}$$

so we continue the above calculation

$$\begin{aligned} & \mathbb{E} \left[\sum_j (|X|u)_{T^j} \chi_{E_j} |Z|w \right] \\ & \leq Q_p(w)^{\max(1, 1/(p-1))} \mathbb{E} \left[\sum_j |X|_{T^j,u} |Z|_{T^j,w} \chi_{E_j} u_{T^j}^{1-p^*/p} w_{T^j}^{1-p^*/p'} \right]_{T^j}. \end{aligned}$$

To continue, step (1) above changes in that $\chi_{S_k^j} = u^{1/p'} w^{1/p} \chi_{S_k^j}$. In the lines that follow, use Hoelder's inequality first and then use Doob's inequality in L^p and $L^{p'}$ to obtain the desired estimate as before. The final norm estimate is a product of the constant in the sparse domination, 8, the constant 2 arising in the steps before (1) and a product of the constants in Doob's inequality $\frac{p}{p-1} \frac{p'}{p'-1}$.

4 Maximal Function of X

In this section we prove Theorem 2.

We use a domination trick for the maximal function by Lerner [13] that gives the correct growth for all p .

$$\begin{aligned} |X_t|^{p-1} & \leq (\mathbb{E}[|X| |\mathcal{F}_t]|)^{p-1} \\ & = (\mathbb{E}[|X| u^{-1} u | \mathcal{F}_t]|)^{p-1} \\ & = (\mathbb{E}_u[|X| u^{-1} | \mathcal{F}_t]|)^{p-1} (\mathbb{E}[u | \mathcal{F}_t]|)^{p-1} \\ & \leq Q_p(w) (\mathbb{E}[w | \mathcal{F}_t])^{-1} (\mathbb{E}_u[|X| u^{-1} | \mathcal{F}_t]|)^{p-1} \end{aligned}$$

Now use that

$$\begin{aligned} (\mathbb{E}_u[|X| u^{-1} | \mathcal{F}_t]|)^{p-1} & = \mathbb{E}[(\mathbb{E}_u[|X| u^{-1} | \mathcal{F}_t]|)^{p-1} | \mathcal{F}_t] \\ & \leq \mathbb{E}[(((X|u^{-1})_u^*)^{p-1} w^{-1} w | \mathcal{F}_t)]. \end{aligned}$$

Then get

$$|X_t|^{p-1} \leq Q_p(w) (((X|u^{-1})_u^*)^{p-1} w^{-1} w)_w^*$$

Thus

$$\begin{aligned}
\mathbb{E}[(X^*)^p w] &\leq Q_p(w)^{p/p-1} \mathbb{E}[(((|X|u^{-1})_u^*)^{p-1} w^{-1})_w^*)^{p'} w] \\
&= Q_p(w)^{p/p-1} \mathbb{E}_w[(((|X|u^{-1})_u^*)^{p-1} w^{-1})_w^*)^{p'}] \mathbb{E}[w] \\
&\leq Q_p(w)^{p/p-1} \left(\frac{p'}{p'-1}\right)^{p'} \mathbb{E}_w[(((|X|u^{-1})_u^*)^{p-1} w^{-1})^{p'}] \mathbb{E}[w] \\
&= Q_p(w)^{p/p-1} \left(\frac{p'}{p'-1}\right)^{p'} \mathbb{E}[(((|X|u^{-1})_u^*)^{p-1} w^{-1})^{p'} w] \\
&= Q_p(w)^{p/p-1} \left(\frac{p'}{p'-1}\right)^{p'} \mathbb{E}[((|X|u^{-1})_u^*)^p u] \\
&= Q_p(w)^{p/p-1} \left(\frac{p'}{p'-1}\right)^{p'} \mathbb{E}_u[((|X|u^{-1})_u^*)^p] \mathbb{E}[u] \\
&\leq Q_p(w)^{p/p-1} \left(\frac{p'}{p'-1}\right)^{p'} \left(\frac{p}{p-1}\right)^p \mathbb{E}_u[(|X|u^{-1})^p] \mathbb{E}[u] \\
&= Q_p(w)^{p/p-1} \left(\frac{p'}{p'-1}\right)^{p'} \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X|u^{-1})^p u] \\
&= Q_p(w)^{p/p-1} \left(\frac{p'}{p'-1}\right)^{p'} \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X|^p w],
\end{aligned}$$

finishing the estimate.

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